ASMR problems

January 2024

1 Geo

Let ABC be a triangle with $\angle ABC = 120^\circ$. Let A', B', C' be the intersections of the internal angle bisectors from A, B, C with sides BC, AC, AB respectively. Determine $\angle A'B'C'$.

2 Algebra

Let $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ be nonnegative real numbers. Prove that

$$\sum_{i,j=1}^{n} \min\{a_i a_j, b_i b_j\} \leq \sum_{i,j=1}^{n} \min\{a_i b_j, a_j b_i\}.$$

3 Combi

Determine all natural numbers n such that one can dissect a square into exactly n acute triangles.

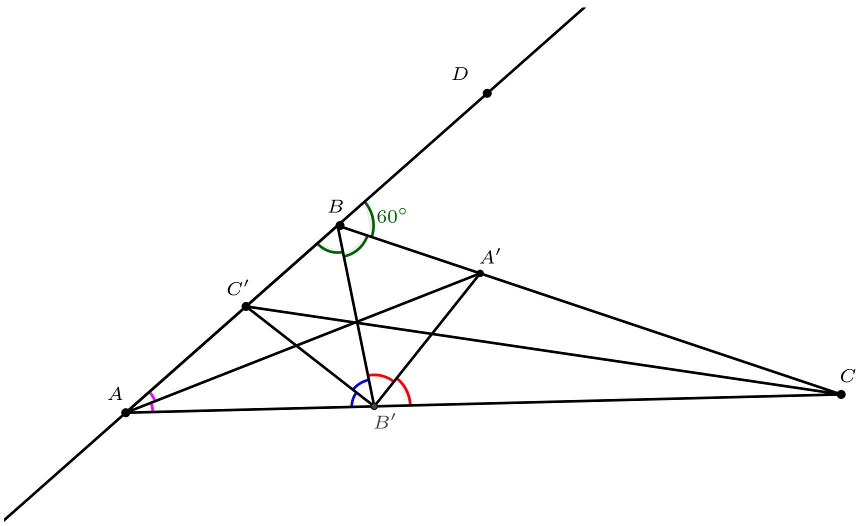
4 NT

Flexi has a secret natural number x which Rösti is trying to discover. At each stage Rösti may only ask questions of the form "is $x + n$ a prime number?" for some natural number n of his choice. Prove that Rösti may discover x using finitely many questions.

5 solutions:

5.1 Geo

5.1.1 Solution



Let D be the a point on the line AB such that $\angle CBD = 60°$, since $\angle ABC = 120°$. Observe BA' is the angle bisector of $\angle B'BD$.

Since AA' and BA' are both angle bisectors they intersect at the center of the ex-circle of $\triangle AB'B$ over the edge $B'B$ along with the third angle bisector $B'A'$.

Therefore $B'A'$ is the angle bisector of $\angle BB'C$ and similarly $B'C'$ is the angle bisector of $\angle AB'B$.

$$\angle CB'A' = \angle A'B'B \text{ and } \angle AB'C' = \angle C'B'B$$

$$\angle CB'A' + \angle AB'C' = \angle C'B'B + \angle BB'A' = \frac{180}{2}$$

$\implies \angle A'B'C' = 90°$

5.1.2 Marking Scheme

Non-additive points

(1.5 points) State BA' or BC is the angle bisector of $\angle B'BD$.

(3 points): State that AA' and BA' both are angle bisectors and intersect in A'.

(7 points): Show that $B'A'$ the angle bisector of $\angle CB'B$

(8 points): Show that both $B'A'$ and $B'C'$ are angle bisectors.

(10.5 points): Finish

If $<$ 7 **points**, the following additional **additive** points are available:

(1 points): Claim the answer is $90°$

(1 points): Claim that $B'A'$ the angle bisector of $\angle CB'B$

5.2 Algebra

5.2.1 first solution

Let us define $\mathbb{1}_c$ for $c \in \mathbb{R}_{\geq 0}$ as the characteristic function of interval $[0, c]$, in other words:

$$\mathbb{1}_c(x) = \begin{cases} 1 \text{ for } x \in [0, c] \\ 0 \text{ for } x \notin [0, c] \end{cases}$$

Lemma 1.

$$\min\{a, b\} = \int_0^\infty \mathbb{1}_a(x)\mathbb{1}_b(x)dx = \int_0^\infty c \cdot \mathbb{1}_{a/c}(x)\mathbb{1}_{b/c}(x)dx$$

Proof: $\mathbb{1}_a(x)\mathbb{1}_b(x) = \mathbb{1}_{\min\{a,b\}}(x)$.
Note that:

$$\sum_{i,j=1}^n \min\{a_ib_j, a_jb_i\} = \sum_{i,j=1}^n \int_0^\infty \mathbb{1}_{a_ib_j}(x)\,\mathbb{1}_{a_jb_i}(x)\mathrm{d}x$$

$$= \sum_{i,j=1}^n \int_0^\infty b_ib_j\mathbb{1}_{a_i/b_i}(x)\,\mathbb{1}_{a_j/b_j}(x)\mathrm{d}x$$

$$= \int_0^\infty \sum_{i,j=1}^n b_i\mathbb{1}_{a_i/b_i}(x)\,b_j\mathbb{1}_{a_j/b_j}(x)\mathrm{d}x$$

$$= \int_0^\infty \left(\sum_{i=1}^n b_i\mathbb{1}_{a_i/b_i}(x)\right)^2 \mathrm{d}x \tag{1}$$

$$= \int_0^\infty \frac{1}{2}\left(\left(\sum_{i=1}^n b_i\mathbb{1}_{a_i/b_i}(x)\right)^2 + \left(\sum_{i=1}^n a_i\mathbb{1}_{b_i/a_i}(x)\right)^2\right)\mathrm{d}x \tag{2}$$

$$\geq \int_0^\infty \left(\sum_{i=1}^n b_i\mathbb{1}_{a_i/b_i}(x)\right)\left(\sum_{i=1}^n a_i\mathbb{1}_{b_i/a_i}(x)\right)\mathrm{d}x \text{ by } AM - GM\text{, monotonicity of integral} \tag{3}$$

$$= \int_0^\infty \sum_{i,j=1}^n b_ia_j\mathbb{1}_{a_i/b_i}(x)\,\mathbb{1}_{b_j/a_j}(x)\mathrm{d}x \tag{4}$$

$$= \int_0^\infty \sum_{i,j=1}^n \mathbb{1}_{a_ia_j}(x)\,\mathbb{1}_{b_ib_j}(x)dx$$

$$= \sum_{i,j=1}^n \int_0^\infty \mathbb{1}_{a_ia_j}(x)\,\mathbb{1}_{b_ib_j}(x)dx = \sum_{i,j=1}^n \min\{a_ia_j, b_ib_j\} \tag{5}$$

which proves the inequality.

5.2.2 marking scheme:

The following is not additive:
Defining some function similar to $\mathbb{1}_c(x)$: 2.5 Pts.
Stating and proving something along the lines of Lemma 1: 5.5 points.
Getting to step (n) of the solution: $5.5 + n$ points.
Having a complete solution: 10.5 points.

5.3 Combi

The answer is that it is possible if and only if $n \geq 8$. We proceed as follows:

- Prove that $n \leq 7$ does not work
- Construct solutions for $n = 8, 9, 10$
- Do induction to create a construction for $n + 3$ starting from n

5.3.1 Proving that $n \leq 7$ is impossible (sketch)

We define a "cutting edge"(CE) to be an edge that has both vertices on the sides of the square. We first claim that there is no cutting edge.

Lemma. *If there is a cutting edge, $n \geq 8$*

Proof. If there is a cutting edge, one of the following 4 cases can appear:

1. The CE divides the square into two non acute triangles (NAT)
2. The CE divides the square into an NAT and a quadrilateral with 3 non acute angles (3NAQ)
3. The CE divides the square into an NAT and a pentagon
4. The CE divides the square into two 3NAQ

We claim that we need at least 5 acute triangles to dissect an NAT and at least 4 to dissect a 3NAQ. We need at least 3 triangles to dissect a pentagon (by normal triangulation). This proves that in this case, $n \geq 8$.

NAT: let $\triangle ABC$ such that the interior angle at A is obtuse, with the assumption that the amount of acute triangles needed to dissect $\triangle ABC$ is minimal. It follows that there is one point P in the interior that is connected with A. if viewed as a graph, $deg(P) \geq 4$. P is at most connected to one of B or C (else it would be connected to A, B, C creating another obtuse triangle, contradicting minimality). Thus we have 2 more points R, Q in this configuration that are connected with P. every vertex other than A, B, C has degree ≥ 4 (if viewed in the graph).
vertex A has degree ≥ 3
vertices B, C have degree ≥ 2
The amount of points $|P|$ other than A, B, C is ≥ 3.
Thus, the amount of edges $E \geq 1/2(3 + 2 + 2 + 4|P|)$
The amount of vertices $V = 3 + |P|$ By the euler characteristic of plane graphs, we have $F = 2+E-V \geq 2+7/2+2|P|-3-|P| = 2.5+|P| \geq 5.5$. one face is the outside, thus at least 5 triangles is guaranteed

3NAQ: at least 3 have deg 3, meaning at least one interior point, this interior point must have deg ≥ 4 implying at least 4 triangles.

□

Definition. *To proceed we now define different types of points.*

- E-points *are points on the edges of the square.*
- T-points *are interior points of degree* 4. *Note that these points always lie on a triangle edge, and have 3 triangles on the other side (proof: trivial angle sums)*
- B-points *are interior points of degree at least* 5.

e, t and b are the amounts of these points

If there is no cutting edge and no B-point, then there are at least 3 T-points. *Proof:* start at a corner and follow the line segment until where it ends. That ending point is on another line segment and is our first t-point. Follow the next line segment to its end, boom, second t-point. Follow the next line segment to its end, boom, third t-point (this will not lead to first t point because else there is a cutting edge). now do some case distinctions and you are done (there are like 10 cases but they are all somewhat easy, totally not coping)

To conclude this case, we can apply triple-counting to get

$$3n \geq 2 \cdot 4 + 3e + 3t$$

If the RHS is 22 or more, we can conclude $n \geq 8$. So assume that

$$21 \geq 8 + 3e + 3t \quad \Rightarrow \quad 4 \geq e + t$$

We basically have two cases: $t = 3$, or $t = 4$

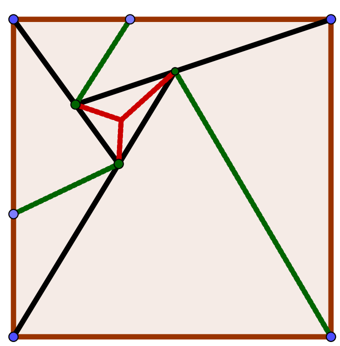


Figure 1: $t = 3, b = 0$. The added edges cannot meet inside the small triangle, for if they did they would either create a B-point or a T-point which would have to be a new edge, requiring the creation of one more T-point (overshooting the limit of 4), so they have to go to the edges. This creates at least two E-points

In the $t = 4$ case it's even clearer that it cannot work without one more point.

If there is a b-point, note that it is connected to at most 3 of the 4 vertices. By triple counting (and seeing that each corner has at least 2 triangles in it)

$$3n \geq 2 \cdot 4 + 1 \cdot 5 + 3e + 3t$$

If the RHS is at least 22, then n is at least 8. So say that the RHS is at most 21, that is

$$21 \geq 13 + 3e + 3t \quad \Leftrightarrow \quad 8 \geq 3e + 3t$$

Clearly $e \geq 1$ (else draw four semicircles over the sides, contradiction), so there is at most one T-point. No T-point would imply that there is no interior point besides the one B-point clearly cannot be the case, therefore there must be a T-point as well. Note that in this case the b point must be connected to 3 of the corners, else we get $e + t \geq 3$, contradiction. But then the five edges going out from the B-point must end in the three corners and the E-point and the T-point. Thus the B-point cannot be the end of the triangle edge on which the T-point is, therefore there is some other line going through the T-point. But that line cannot go through the E-point and the missing corner at the same time, since that line would either create a new intersection or it would overlap with the side of the square. Either way we get that there must be some other point, hence $e + t \geq 3$, contradiction.

5.3.2 Constructing $n = 8, 9, 10$

It should be obvious how to do this from the figures below.

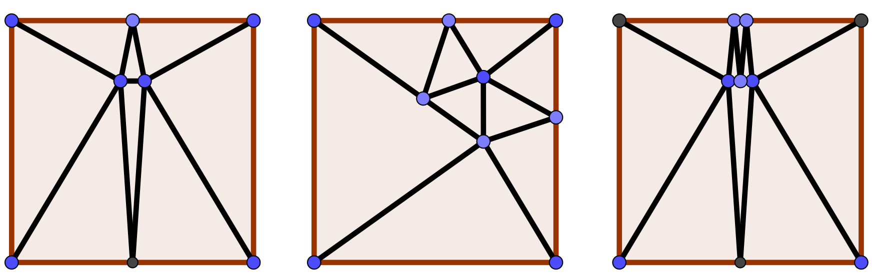


Figure 2: $n = 8, 9, 10$

5.3.3 Induction

Assuming we have a construction for n, we can take any of its triangles and subdivide it using its medial triangle. This gives a valid solution for $n + 3$ since each of the new triangle is similar to the previous one and thus acute. Starting from $n = 8, 9, 10$, we can get to all $n \geq 8$. Since we proved that $n \leq 7$ is impossible, these are the only solutions.

5.3.4 Marking Scheme (additive)

6 points: proof that $n \leq 7$ doesn't work (the gory details will be graded individually).
4 points: construction for $n \geq 8$:

- $3 \cdot 0.5$ points: constructions for $n = 8, 9, 10$
- 2.5 points: medial triangle induction

0.5 points: finish

5.4 NT

The main idea is to use a fixed number of questions for each $n \in \mathbb{N}$ that can determine whether $x = n$ or not. If this is possible for every $n \in \mathbb{N}$, we can find x by simply confirming for every $n \in \mathbb{N}$ whether $x = n$ or not, starting with $n = 1$, increasing n by 1 in each step. Since x is finite, this process ends after a finite amount of steps.
An algorithm that finds out whether $x = n$ or not in $< 2n$ steps:
Let p be a prime $> n$. Let us define $a_{n,0} = p - n$. By Dirichlet, there exists infinitely many primes of the form $ap + b \forall b \in \{1, \dots, p-1\}$. For every $b \in \{1, \dots p-1\}$ pick a prime $q_b > 2p$ such that $q_b \equiv b \mod p$. Define $a_{n,b} = q_b - n$. Assume that the question " is $x + a_{n,k}$ a prime?" is answered with yes for all $k \in \{0, \dots, p-1\}$. Since

$$\{x + a_{n,k} \mod p \,|\, 0 \le k \le p-1\} = \{0, \dots, p-1\},$$

at least one of the $x + a_{n,k}$ is divisible by p. From there it follows that $\exists k : x + a_{n,k} = p$, but note that $a_{n,k} > p \,\forall\, k \ge 1$. Therefore $x + a_{n,0} = p$, implying $x = p - a_{n,0} = n$. By asking the question " is $x + a_{n,k}$ a prime?" for $0 \le k \le p-1$ we can therefore determine whether $x = n$ or not. Thus, we have our desired algorithm that can determine whether $x = n$ in a fixed amount of steps.

5.4.1 marking scheme

The following are non-additive:
Finding an algorithm that determines whether $x = c$ for a concrete number: up to 3 points. (let's say 1.5 points per c)
Finding an algorithm that works for all $n \in \mathbb{N}$: 9 points
Complete solution: 10.5 points
small errors: penalty up to 3 points (depending on error)